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LETTER TO THE EDITOR

Nature of the transition in non-linear spin-S Ising models (integer spin)

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Abstract. We consider the spin-S Hamiltonian

$$-\beta\mathscr{H}=K\sum_{\langle i,j\rangle}Q_S(S_i)Q_S(S_j)+\Lambda\sum_k Q_S(S_k)$$

where $Q_S(y)$ is the polynomial in y of degree $2S$ whose eigenvalues are ± 1. Denoting the partition function by $Z_S(K,\Lambda)$, it is shown rigorously that

$$Z_S(K,\Lambda)=[S(S+1)]^{\frac{1}{2}N}Z_{\frac{1}{2}}(K,\Lambda+C),\qquad 2C=\ln(1+S^{-1}),$$

for all integer values of S.

Recently (Joseph 1976a) we considered the spin-S Hamiltonian

$$\mathscr{H}=-J\sum_{\langle i,j\rangle}Q_S(S_i)Q_S(S_j)-\lambda\sum_{k=1}^{N}Q_S(S_k) \tag{1}$$

where $Q_S(y)$† is the polynomial in y of degree $2S$ whose eigenvalues are just ± 1, and proved for all half-integer values of S that the partition function $Z_S(K,\Lambda)$, $K=\beta J$, $\Lambda=\beta\lambda$, differed from that for $S=\frac{1}{2}$ by a simple multiplicative constant. Comparing the structure of the interaction operator $Q_S(S_i)Q_S(S_j)$ (Joseph 1976b) to that appropriate to the $(2S+1)$-component Potts model, $\delta_{S_iS_j}$ (Joseph and Kim 1974), we concluded that the relative changes in terms from the later to the former was sufficient for half-integer values of S to change any discontinuous transition found for the Potts model (Baxter 1973, Kim and Joseph 1975) into a continuous one. In the present Letter we consider the case of integer values of S and show that $Z_S(K,\Lambda)\propto Z_{\frac{1}{2}}(K,\Lambda+C)$ where $C=\ln[1+S^{-1}]^{1/2}$. Hence, for $\Lambda=0$, the QQ model is equivalent to a spin-$\frac{1}{2}$ Ising model in a temperature-dependent 'field' and has no transition at all‡. The relevance of this result to the Potts model can be appreciated by comparing the forms of the Hamiltonians; for $S=1$ for example

$$\begin{aligned}\mathscr{H}_{ij}^{(QQ)}&=4S_i^2S_j^2-2(S_i^2+S_j^2)+1,\\ 2\mathscr{H}_{ij}^{(\text{Potts})}-1&=S_iS_j+3S_i^2S_j^2-2(S_i^2+S_j^2)+1.\end{aligned} \tag{2}$$

Hence a change in the numerical values of the terms in $H_{ij}^{(\text{Potts})}$ to those of $H_{ij}^{(QQ)}$ is

† $Q_{1/2}(y)=2y$, $Q_1(y)=2y^2-1$, $Q_{3/2}(y)=\frac{4}{3}y^3-\frac{7}{3}y$, $Q_2(y)=\frac{2}{3}y^4-\frac{8}{3}y^2+1,\ldots$.
‡ If $0<-\lambda<kT_cC$, there is a discontinuous transition at $kT=-\lambda/C$; T_c is the transition temperature for $S=\frac{1}{2}$. If $-\lambda=kT_c/C$, the transition is a continuous one at T_c. These results apply in both two and three dimensions.

sufficient to destroy any transition that occurs in the Potts model (Baxter 1973, Kim and Joseph 1975).

Consider then the Hamiltonian given in equation (1). Since the eigenvalues of $Q_S(S_i)$ are just $(-1)^{2S+n_i-1}$ (Joseph 1976b), $n=1,2,\ldots,2S+1$, the partition function can be written in the form

$$Z_S(K,\Lambda)=(\cosh K)^{\mathscr{P}}\,\mathrm{Tr}\prod_{\langle i,j\rangle}[1+v(-1)^{n_i+n_j}]\prod_k \exp[\Lambda(-1)^{2S+n_k-1}] \tag{3}$$

where $\mathscr{P}=\frac{1}{2}Nz$, $v=\tanh K$, z being the lattice coordination number. It is convenient at this point to introduce the quantities

$$A_S\equiv\sum_{n=1}^{2S+1}\exp[\Lambda(-1)^{2S+n-1}]=\begin{cases}(2S+1)\cosh\Lambda, & \text{half-integer } S\\ 2[S(S+1)]^{1/2}\cosh(\Lambda+C), & \text{integer } S\end{cases} \tag{4}$$

and

$$B_S\equiv\sum_{n=1}^{2S+1}(-1)^n\exp[\Lambda(-1)^{2S+n-1}]=\begin{cases}(2S+1)\sinh\Lambda, & \text{half-integer } S\\ -2[S(S+1]^{1/2}\sinh(\Lambda+C), & \text{integer } S\end{cases} \tag{5}$$

where $C\equiv\frac{1}{2}\ln(1+S^{-1})$. The factor multiplying $(\cosh K)^{\mathscr{P}}$ on the right-hand side of equation (3) can be considered as a power series in v. In this expansion, the first term is just A_S^N and each l line graph which can be drawn on a lattice contributes to the lth power of v with a factor determined by the topology of the graph. Each graph of k vertices, m of which are odd (that is, has an odd number of lines meeting at this point), contributes a factor $A_S^{N-k}B_S^mA_S^{k-m}=A_S^N[B_S/A_S]^m$; m is always zero or an even integer. Factoring out the term A_S^N, $Z_S(K,\Lambda)$ is then of the form

$$Z_S(K,\Lambda)=(\cosh K)^{\mathscr{P}}A_S^N\mathscr{Z}(v,|B_S/A_S|); \tag{6}$$

the only S-dependence in $\mathscr{Z}$ occurs by virtue of the quantity B_S/A_S. For S a half-integer, equation (6) becomes on using equations (4) and (5),

$$Z_S(K,\Lambda)=(S+\tfrac{1}{2})^N(\cosh K)^{\mathscr{P}}[2\cosh\Lambda]^N\mathscr{Z}(v,|\tanh\Lambda|)=(S+\tfrac{1}{2})^NZ_{1/2}(K,\Lambda), \tag{7}$$

in agreement with Joseph (1976a). For S an integer, we get similarly

$$\begin{aligned}Z_S(K,\Lambda)&=[S(S+1)]^{\frac{1}{2}N}(\cosh K)^{\mathscr{P}}[2\cosh(\Lambda+C)]^N\mathscr{Z}(v,|\tanh(\Lambda+C)|)\\&=[S(S+1)]^{\frac{1}{2}N}Z_{1/2}(K,\Lambda+C).\end{aligned} \tag{8}$$

Note that for $S\to\infty$, $C\to 0$ and $Z_S(K,\Lambda)\to Z_{1/2}(K,\Lambda)$.

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